First-degree birational transformations of the Painlevé equations and their contiguity relations

2001 J. Phys. A: Math. Gen. 34 10507

(http://iopscience.iop.org/0305-4470/34/48/315)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10507–10522 PII: S0305-4470(01)25340-2

First-degree birational transformations of the Painlevé equations and their contiguity relations

Robert Conte[1,3] **and Micheline Musette**[2]

[1] Service de physique de l'état condensé, CEA-Saclay, F-91191 Gif-sur-Yvette Cedex, France
[2] Dienst Theoretische Natuurkunde, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail: Conte@drecam.saclay.cea.fr and MMusette@vub.ac.be

Received 30 May 2001, in final form 3 October 2001
Published 23 November 2001
Online at stacks.iop.org/JPhysA/34/10507

Abstract

We present a consistent truncation, allowing us to obtain the first-degree birational transformation found by Okamoto for the sixth Painlevé equation. The discrete equation arising from its contiguity relation is then just the sum of six simple poles. An algebraic solution is presented, which is equivalent to but simpler than the Umemura solution. Finally, the well known confluence provides a unified picture of all first-degree birational transformations for the lower Painlevé equations, ranging them in two distinct sequences.

PACS numbers: 02.30.+g, 02.30.Hq

1. Introduction

The purpose of this paper is to derive *birational transformations* for the Painlevé equations Pn by a direct method, only based on the singularity structure of the ordinary differential equation (ODE). By definition, a birational transformation is a set of two relations,

$$u = f(U', U, X) \qquad U = F(u', u, x) \tag{1}$$

with f and F rational functions, which maps an equation

$$E(u) \equiv \mathrm{P}n(u, x, \boldsymbol{\alpha}) = 0 \qquad \boldsymbol{\alpha} = (\alpha, \beta, \gamma, \delta) \tag{2}$$

into the same equation with different parameters

$$E(U) \equiv \mathrm{P}n(U, X, \boldsymbol{A}) = 0 \qquad \boldsymbol{A} = (A, B, \Gamma, \Delta) \tag{3}$$

with some homography (usually the identity) between x and X. The parameters $(\boldsymbol{\alpha}, \boldsymbol{A})$ must obey as many algebraic relations as elements in $\boldsymbol{\alpha}$. The *degree* of a birational transformation is defined as the highest degree in U' or u' of the numerator and the denominator of (1).

[3] Author for correspondence.

Since the sixth of these Painlevé equations Pn generates the five others by a confluence process [1], we first concentrate on it

$$\begin{aligned}\text{P6}: u'' &= \frac{1}{2}\left[\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right]u'^2-\left[\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right]u'\\ &\quad+\frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha+\beta\frac{x}{u^2}+\gamma\frac{x-1}{(u-1)^2}+\delta\frac{x(x-1)}{(u-x)^2}\right].\end{aligned}$$

Schlesinger [2, p 144] was the first to prove the existence of a transformation breaking the invariance under permutation of the four singular points $\infty, 0, 1, x$. To express it, one needs to introduce the *monodromy exponents* (according to the classical terminology [3]), i.e. the four components of the vectors

$$\boldsymbol{\theta}=\begin{pmatrix}\theta_\infty\\ \theta_0\\ \theta_1\\ \theta_x\end{pmatrix}\qquad \boldsymbol{\Theta}=\begin{pmatrix}\Theta_\infty\\ \Theta_0\\ \Theta_1\\ \Theta_x\end{pmatrix} \tag{4}$$

only defined by their squares,

$$\theta_\infty^2=2\alpha\qquad \theta_0^2=-2\beta\qquad \theta_1^2=2\gamma\qquad \theta_x^2=1-2\delta \tag{5}$$

$$\Theta_\infty^2=2A\qquad \Theta_0^2=-2B\qquad \Theta_1^2=2\Gamma\qquad \Theta_x^2=1-2\Delta. \tag{6}$$

The transformation found by Schlesinger conserves two monodromy exponents and shifts the two others by one unit, up to sign changes of course,

$$\begin{aligned}&\theta_i=\Theta_i\qquad \theta_j=\Theta_j\qquad \theta_k=\Theta_k+1\qquad \theta_l=\Theta_l+1\\ &(i,j,k,l)\ \text{permutation of}\ (\infty,0,1,x).\end{aligned} \tag{7}$$

However, Schlesinger did not give the associated birational representation.

This was achieved by Garnier [4,5], to establish the missing proof of a theorem of Schwarz on the problem of Plateau. The birational transformation which realizes the shifts (7) is of second degree,

$$\frac{uU}{x}=\frac{R_n^+R_n^-}{R_d^+R_d^-}\qquad x=X \tag{8}$$

$$R_n^\pm=\frac{x(x-1)U'}{U(U-1)(U-x)}+\frac{\pm\Theta_0}{U}+\frac{\Theta_1}{U-1}+\frac{\Theta_x-1}{U-x} \tag{9}$$

$$R_d^\pm=\frac{x(x-1)U'}{U(U-1)(U-x)}+\frac{\pm\Theta_\infty-\Theta_1-\Theta_x+1}{U}+\frac{\Theta_1}{U-1}+\frac{\Theta_x-1}{U-x} \tag{10}$$

$$\text{T}_\text{G}:\boldsymbol{\theta}=\begin{pmatrix}\Theta_\infty\\ \Theta_0\\ -\Theta_1\\ -\Theta_x\end{pmatrix}+\begin{pmatrix}0\\0\\1\\1\end{pmatrix}. \tag{11}$$

Since the transformation between $\boldsymbol{\theta}$ and $\boldsymbol{\Theta}$ is an involution, the inverse of (8) is readily obtained by exchanging $(u,\boldsymbol{\theta})$ and $(U,\boldsymbol{\Theta})$. We will adopt such a convention (choice of signs so as to have involutions) throughout this paper.

This birational transformation was later rediscovered by several authors [6–8]. Garnier added that the relation (8) can be considered as a generalization of the *contiguity relation* for P6, thus extending the well known similar notion introduced by Gauss for the hypergeometric equation. However, (8) is not a 'pure' contiguity relation, but a two-point relation also involving derivatives. A three-point relation without derivatives can be found in section 7.

Another, elegant way of finding birational transformations has been devised by Okamoto [7]. The Hamiltonian of P6 found by Malmquist [9] has no such invariance as (7), but a quite simple (in fact, affine) transformation makes it invariant under any permutation of four components b_j different from (but equivalent to) θ_j, defined as

$$\begin{aligned} b_1 &= (\theta_0 + \theta_1)/2 \qquad & b_2 &= (\theta_0 - \theta_1)/2 \\ b_3 &= (\theta_x - 1 - \theta_\infty)/2 \qquad & b_4 &= (\theta_x - 1 + \theta_\infty)/2. \end{aligned} \tag{12}$$

This allowed Okamoto to build various birational transformations, by construction canonical in the Hamiltonian sense. One of them (his 'parallel transformation') is identical to that of Garnier. Another one (denoted $w_1 w_2 w_1$ in [7, p 356 example 2.1] and T_6 here) seems to have degree two but in fact has degree one. Indeed, the first line of the matrix equation which defines it is

$$q_w = q + \frac{(b_3 - b_1) q (q-1) p}{\mathrm{d}h/\mathrm{d}t + b_1^2} \tag{13}$$

with the correspondence of notation $q_w = u, q = U$ and p linear in our U', and, with the expression for $\mathrm{d}h/\mathrm{d}t$ on the last line of p 356 (in which a factor q has been omitted, see the P6 Hamiltonian p 339, repeated p 348), a common factor p to the numerator and the denominator cancels out,

$$q_w = q + \frac{(b_3 - b_1) q (q-1)}{-q(q-1)p + [b_1(2q-1) - b_2]} \tag{14}$$

yielding a first-degree transformation. This transformation is more elementary in the sense that the transformation of Garnier is an integer power [10] of T_6. This transformation T_6 has recently been rediscovered [11], up to a simultaneous homography on (u, x).

To achieve our goal (rely only on the singularity structure to find birational transformations), we need to improve the *singular manifold method* so that it succeeds in obtaining a birational transformation for P6 and, by confluence, for each Pn equation, $n = 5, 4, 3, 2$ (P1 depends on no parameter and thus admits no birational transformation). Originally introduced for partial differential equations by Weiss *et al* [12], the singular manifold method is a powerful tool for deriving Bäcklund transformations, by considering only the singularity structure of the solutions. Its current achievements are detailed in summer school proceedings, see [13, 14]. An extension to ODEs has been proposed [15, 16] to derive a birational transformation for the Painlevé equations, but its application to the master equation P6 is still an open problem. We solve it here by implementing an essential piece of information, which has up to now been overlooked.

The paper is organized as follows. In section 2, we exploit the information that there always exists a homography between the derivative of the solution of the considered Painlevé equation Pn and the Riccati pseudopotential Z introduced in the 'truncation' assumption. This reduces the problem to finding two functions of two variables instead of two functions of three variables.

In section 3, we implement this homography in the one-family truncation, which allows us to overcome the major difficulty, coming, in the case of P6, from the value 1 of the Fuchs index. We also improve a previous conjecture [17] on the necessary form of the birational transformation. The truncation then becomes easy to solve and, up to the four homographies on (U, x) which conserve x, it admits a solution, identical to the transformation $w_1 w_2 w_1$ of Okamoto.

In section 4, we give the various representations of this transformation.

In section 5, by looking for the fixed points of the birational transformations of P6, we obtain a quite simple algebraic solution.

In section 6, starting from the birational transformation of P6, we perform the classical confluence, and obtain two sequences of first-degree birational transformations for the lower Pn equations.

In section 7, we solve the recurrence relation between the monodromy exponents of P6, and we build the contiguity relation, depending on four arbitrary parameters.

In [18] we have already presented several results, in particular the detailed algorithm of the truncation (which we do not repeat here) and its result for P6. What is new here is the first coherent description of all the first-degree birational transformations of all the Pn equations, obtained by the simultaneous application of two noncommuting operations, the confluence from Pn to a lower Pn and the homographies conserving Pn. This unified picture in turn generates a similar unification of all the contiguity relations (discrete equations) which arise from a birational transformation.

Throughout the paper, we consider as identical two birational transformations which only differ by sign changes and homographies. Moreover, without loss of generality, our birational transformations are always involutions for the same, consistent, choice of signs.

2. The fundamental homography

From the result of Fuchs [3], the P6 equation is obtained from the zero-curvature condition of a linear differential system of order two. Therefore the pseudopotential of the singular manifold method has only one component [19], which can be chosen so as to satisfy some Riccati ODE.

Each Pn equation which admits a birational transformation has one or several (four for P6) couples of families of movable simple poles with opposite residues $\pm u_0$, therefore both the one-family truncation and the two-family truncation are applicable. In this paper, we consider only the one-family truncation, whose assumption is [16]

$$u = u_0 Z^{-1} + U \qquad u_0 \neq 0 \qquad x = X \tag{15}$$

$$Z' = 1 + z_1 Z + z_2 Z^2 \qquad z_2 \neq 0 \tag{16}$$

in which u and U satisfy (2) and (3), (Z, z_1, z_2) are rational functions of (x, U, U') to be determined. After this is done, the relation (15) represents half of the birational transformation.

Besides the equation (16), there exists a second Riccati equation in the present problem. This is the Painlevé equation (3) itself. Indeed, any Nth order, first-degree ODE with the Painlevé property is necessarily [20, pp 396–409] a Riccati equation for $U^{(N-1)}$, with coefficients depending on x and the lower derivatives of U, in our case

$$U'' = A_2(U, x)U'^2 + A_1(U, x)U' + A_0(U, x). \tag{17}$$

Since the group of invariance of a Riccati equation is the homographic group, the variables U' and Z are linked by a homography, the three coefficients g_j of which are rational in (U, x). Let us define it as

$$(U' + g_2)(Z^{-1} - g_1) - g_0 = 0 \qquad g_0 \neq 0. \tag{18}$$

This allows us to obtain the two coefficients z_j of the Riccati pseudopotential equation (16) as explicit expressions of $(g_j, \partial_U g_j, \partial_x g_j, A_2, A_1, A_0, U')$. Indeed, eliminating U' between (17) and (18) defines a first-order ODE for Z, whose identification with (16) *modulo* (18) provides three relations.

For the one-family truncation, these are

$$g_0 = g_2^2 A_2 - g_2 A_1 + A_0 + \partial_x g_2 - g_2 \partial_U g_2 \tag{19}$$

$$z_1 = A_1 - 2g_1 + \partial_U g_2 - \partial_x \mathrm{Log}\, g_0 + (2A_2 - \partial_U \mathrm{Log}\, g_0)\, U' \tag{20}$$

$$z_2 = -g_1 z_1 - g_1^2 - g_0 A_2 - \partial_x g_1 - (\partial_U g_1) U'. \tag{21}$$

Therefore, the natural unknowns in the present problem are the two coefficients g_1, g_2 of the homography, which are functions of the two variables (U, x), and not the two functions (z_1, z_2) of the three variables (U', U, x).

Remark. One must also consider the case when the relation between Z^{-1} and U' is affine, excluded in (18). Assuming

$$G_1(U' + G_2) - Z^{-1} = 0 \qquad G_1 \neq 0 \tag{22}$$

the equation analogous to (19) is now

$$\partial_U G_1 + G_1^2 + A_2 G_1 = 0 \tag{23}$$

which for P6 admits no solution G_1 rational in U.

3. The truncation

Just as the field u is represented, see equation (15), by a Laurent series in Z which terminates ('truncated series'), the lhs $E(u)$ of the Pn equation can also be written as a truncated series in Z. This is achieved by the elimination of u, Z', U'', U' between (2), (3), (15), (16) and (18), followed by the elimination of (g_0, z_1, z_2) from (19)–(21) (q denotes the singularity order of Pn written as a differential polynomial in u, it is -6 for P6),

$$E(u) = \sum_{j=0}^{-q+2} E_j(U, x, u_0, g_1, g_2, \boldsymbol{\alpha}, \boldsymbol{A}) Z^{j+q-2} = 0 \tag{24}$$

$$\forall j \ : \ E_j(U, x, u_0, g_1, g_2, \boldsymbol{\alpha}, \boldsymbol{A}) = 0. \tag{25}$$

The nonlinear *determining equations* $E_j = 0$ are independent of U', and this is the main difference with previous work [16]. Another difference is the greater number ($-q + 3$ instead of $-q + 1$) of equations $E_j = 0$, which is due to the additional elimination of U' with (18).

The $-q + 3$ determining equations (25) in the three unknown functions $u_0(x)$, $g_1(U, x)$, $g_2(U, x)$ (and the unknown scalars α, β, γ, δ in terms of A, B, Γ, Δ) must be solved, as usual, by increasing values of their index j.

For any solution $g_j(U, x)$, there exist evidently three other solutions, generated by the action on $g_j(U, x)$ of the three homographies of U which conserve x and P6, namely

$$x = X, \begin{cases} \mathrm{H}_{dcba} \ : \ u - x = \dfrac{x(x-1)}{U - x} \\ \mathrm{H}_{badc} \ : \ u = \dfrac{x}{U} \\ \mathrm{H}_{cdab} \ : \ u - 1 = \dfrac{1 - x}{U - 1}. \end{cases} \tag{26}$$

In order to shorten the resolution, it is advisable to enforce the condition of birationality, which requires that g_k, $k = 0, 1, 2$, be the quotient of two polynomials of U. After their degree has been found, the equations (25) split into an even more overdetermined set of equations involving functions of x only, which is much easier to solve.

But the decisive shortening results from the following stronger condition. From the expression of the direct birational transformation,

$$u = U + u_0 \left(g_1(U, x) + \frac{g_0(U, x)}{U' + g_2(U, x)} \right) \tag{27}$$

the movable values of U which make the field u infinite are defined (apart from the poles arising from the terms U, g_1, and g_0, which are fixed) by the ODE

$$U' + g_2(U, x) = 0. \tag{28}$$

For all known birational transformations of Pn (see the book [21] for P2–P5, [5, formula (2.8)] and [7, p 356] for P6), it happens that the ODE analogous to (28) is a Riccati ODE (or a product of Riccati ODEs in [5]), i.e. the unique first-order first-degree ODE which has the Painlevé property. In at least one other example of higher order [22], the birational transformation between two different ODEs having the Painlevé property has a denominator which defines a P1 equation. Let us conjecture the generality of this property.

Conjecture. *Given a birational transformation between two ODEs having the Painlevé property, the ODEs defined by its denominators also have the Painlevé property.*

This is an improvement of a previous conjecture by Fokas and Ablowitz [17, formula (2.6)] in two respects: both fields u and U are required to satisfy the same ODE, and no specific U-dependence is assumed for g_0 and g_1; their conjecture happens to be true for P2–P5 but not for P6.

The practical resolution for P6 is performed in [18] and, *modulo* the homographies on U which conserve x, one finds the solution

$$g_0 = \frac{NU(U-1)(U-x)}{u_0 x(x-1)} \qquad g_1 = 0$$

$$g_2 = \frac{U(U-1)(U-x)}{x(x-1)}\left(\frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x - 1}{U-x}\right) \tag{29}$$

$$u_0 = -\frac{x(x-1)}{\theta_\infty} \qquad \theta_\infty = \tfrac{1}{2}(\Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x + 1) \tag{30}$$

$$N = 1 - \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x \tag{31}$$

$$z_1 = \frac{1}{x(x-1)}((\Theta_1 + \Theta_x - 1)U + (\Theta_x - 1 + \Theta_0)(U-1) + (\Theta_0 + \Theta_1)(U-x)) + \frac{1}{x} + \frac{1}{x-1} \tag{32}$$

$$z_2 = \frac{N\theta_\infty}{2(x(x-1))^2}((U-1)(U-x) + U(U-x) + U(U-1)). \tag{33}$$

Although, due to lack of time, we have not yet examined all the subcases of this resolution, this solution is quite probably unique.

4. A first-degree birational transformation of P6

Let us denote it T_6. The eight signs s_∞, s_0, s_1, s_x and S_∞, S_0, S_1, S_x, with $s_j^2 = S_j^2 = 1$, of the monodromy exponents remain arbitrary and independent. Whenever there is no ambiguity, T_6 will also denote the case with all +1 signs.

The *affine representation* of T_6 is

$$\mathrm{T}_6 : s_j\theta_j = S_j\Theta_j - \tfrac{1}{2}\left(\sum S_k\Theta_k\right) + \tfrac{1}{2} \tag{34}$$

$$\mathrm{T}_6^{-1} : S_j\Theta_j = s_j\theta_j - \tfrac{1}{2}\left(\sum s_k\theta_k\right) + \tfrac{1}{2} \tag{35}$$

in which $j, k = \infty, 0, 1, x$. The convention adopted for the signs is aimed at making T_6 equal to its inverse when the signs verify $S_j = s_j$.

Denoting N the odd-parity constant

$$N = \sum(\theta_k^2 - \Theta_k^2) \tag{36}$$

$$= 1 - \sum S_k\Theta_k = -1 + \sum s_k\theta_k \tag{37}$$

$$= 2(s_j\theta_j - S_j\Theta_j) \qquad j = \infty, 0, 1, x \tag{38}$$

the *birational representation* is (for clarity, the signs are omitted)

$$\frac{N}{u-U} = \frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x - 1}{U-x} \tag{39}$$

$$= \frac{x(x-1)u'}{u(u-1)(u-x)} + \frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_x - 1}{u-x}. \tag{40}$$

The four algebraic relations between $\alpha, \beta, \gamma, \delta$ and A, B, Γ, Δ, equivalent to (34), are

$$\forall j = \infty, 0, 1, x \; : \; (\theta_j^2 + \Theta_j^2 - (N/2)^2)^2 - (2\theta_j \Theta_j)^2 = 0. \tag{41}$$

5. Fixed points of the birational transformation of P6

Every fixed point of every birational transformation generically defines a zero-parameter algebraic solution in the following way. Given a birational transformation T, its fixed points $\boldsymbol{\theta}$ are by definition all the solutions of the affine matrix equation

$$\mathrm{T}\begin{pmatrix} \theta_\infty \\ \theta_0 \\ \theta_1 \\ \theta_x \end{pmatrix} = \mathrm{P}\begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \\ s_x\theta_x \end{pmatrix} \tag{42}$$

in the unknowns P (a permutation matrix), s_i (four independent signs), θ_i (four complex numbers). The solution u is then the common solution to P6 and to (1) with $u = U$, and therefore it is generically a zero-parameter algebraic solution of P6.

Let us review these fixed points for the three classes of birational transformations: the three homographies on (u, x) (apart the identity) which conserve x, the birational transformation of Garnier, the first-degree birational transformation.

For the three homographies on (u, x) which conserve x, one obtains the zero-parameter algebraic solutions

$$u = \begin{cases} x + \sqrt{x(x-1)} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, \mu^2, \mu^2, \lambda^2) \\ \sqrt{x} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, \lambda^2, \mu^2, \mu^2) \\ 1 + \sqrt{1-x} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, \mu^2, \lambda^2, \mu^2). \end{cases} \tag{43}$$

It is convenient to introduce the crossratio (x_1, x_2, x_3, x_4) of four numbers,

$$(x_1, x_2, x_3, x_4) = \frac{(x_3 - x_1)(x_4 - x_2)}{(x_3 - x_2)(x_4 - x_1)} \tag{44}$$

and to denote a_i, a_j, a_k, a_l an arbitrary permutation of the four singular points $\infty, 0, 1, x$ of P6. Note that $(\infty, 0, 1, z) = z$. The unique expression of the above zero-parameter algebraic solution is then

$$(a_i, a_j, a_k, u) = \sqrt{(a_i, a_j, a_k, a_l)} \qquad \theta_i^2 = \theta_j^2 = \lambda^2 \qquad \theta_k^2 = \theta_l^2 = \mu^2. \tag{45}$$

Curiously, we could not find this quite elementary solution in the classical authors (Painlevé, Boutroux, Garnier, Malmquist). The first mention of its existence seems to be in [23, equation (4.4)].

For the second-degree birational transformation of Garnier, there exists only one fixed point, found by Umemura [24], which satisfies (11) with $\boldsymbol{\Theta} = \boldsymbol{\theta}$. This solution (two components of $\boldsymbol{\theta}$ equal to $\pm 1/2$, the two others arbitrary) corresponds to two zero-parameter algebraic solutions, either (as originally written by Umemura)

$$u = \begin{cases} \dfrac{\lambda^2 x + \lambda\mu\sqrt{x(x-1)}}{\lambda^2 x + \mu^2(1-x)} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(1/4, \lambda^2, \mu^2, 1/4) \\ \dfrac{(\lambda^2-\mu^2)x + \lambda\mu(x-1)\sqrt{x}}{\lambda^2 x - \mu^2} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(1/4, 1/4, \mu^2, \lambda^2) \\ \dfrac{\lambda^2 x - \lambda\mu x\sqrt{1-x}}{\lambda^2 - \mu^2(1-x)} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(1/4, \lambda^2, 1/4, \mu^2) \end{cases} \tag{46}$$

or

$$u = \begin{cases} x - \dfrac{\lambda}{\mu}\sqrt{x(x-1)} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, 1/4, 1/4, \mu^2) \\ -\dfrac{\lambda}{\mu}\sqrt{x} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, \mu^2, 1/4, 1/4) \\ 1 + \dfrac{\mu}{\lambda}\sqrt{1-x} & \boldsymbol{\theta}^2 = {}^{\mathrm{t}}(\lambda^2, 1/4, \mu^2, 1/4) \end{cases} \tag{47}$$

in which (λ, μ) are arbitrary.

For the first-degree birational transformation $\mathrm{T} = \mathrm{T}_6$, the property that $\sum \theta_j^2$ is conserved is equivalent to the property that $\sum s_j \theta_j$ is unity. Therefore, from equation (38), there is a unique fixed point, characterized by the simultaneous vanishing of both sides of (39), this is the set of one-parameter solutions of the Riccati equation, first found by Fuchs.

Under the action of T_6 (*modulo* signs and homographies), the algebraic solution (43) is mapped either to (46) or to (47). These two solutions, equivalent under T_6, are inequivalent under the birational transformation of Garnier.

Since the set (43) and the sets (46), (47) belong to the same equivalence class, it is advisable to choose (43) as the representative of the solution of Umemura.

6. Confluence to first-degree birational transformations

Let us show that *all* first-degree birational transformations of the lower Pn equations can be generated from the first-degree birational transformation T_6 of P6.

Let us first define the Pn equations which admit a birational transformation as

$$\mathrm{P6} \;:\; u'' = \frac{1}{2}\left[\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-x}\right]u'^2 - \left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{u-x}\right]u'$$
$$+ \frac{u(u-1)(u-x)}{x^2(x-1)^2}\left[\alpha + \beta\frac{x}{u^2} + \gamma\frac{x-1}{(u-1)^2} + \delta\frac{x(x-1)}{(u-x)^2}\right]$$

$$\mathrm{P5} \;:\; u'' = \left[\frac{1}{2u} + \frac{1}{u-1}\right]u'^2 - \frac{u'}{x} + \frac{(u-1)^2}{x^2}\left[\alpha u + \frac{\beta}{u}\right] + \gamma\frac{u}{x} + \delta\frac{u(u+1)}{u-1}$$

$$\mathrm{P4} \;:\; u'' = \frac{u'^2}{2u} + \gamma\left(\frac{3}{2}u^3 + 4xu^2 + 2x^2u\right) - 2\alpha u + \frac{\beta}{u}$$

$$\mathrm{P3}' \;:\; u'' = \frac{u'^2}{u} - \frac{u'}{x} + \frac{\alpha u^2 + \gamma u^3}{4x^2} + \frac{\beta}{4x} + \frac{\delta}{4u}$$

$$\mathrm{P2} \;:\; u'' = \delta(2u^3 + xu) + \alpha.$$

As compared to the usual choice [25]

$$\mathrm{P4} \;:\; \gamma = 1$$
$$\mathrm{P3}(u, x, \alpha, \beta, \gamma, \delta) = \mathrm{P3}'(xu, x^2, \alpha, \beta, \gamma, \delta)$$
$$\mathrm{P2} \;:\; \delta = 1$$

Table 1. Definition of the monodromy exponents and other useful data for the Pn equations. We follow the notation of Okamoto [26], rather than the one of Jimbo and Miwa [27], and each monodromy exponent θ_j, including θ_∞, has a square rational in $\alpha, \beta, \gamma, \delta$. The successive lines are: the singularity order q of the Pn equation and the positive Fuchs index i, the value of the first coefficient u_0 of the Laurent series for u, the notation for the square root of u_0, the definition of the monodromy exponents $\boldsymbol{\theta}$, the components of the column vector $\boldsymbol{\theta}$.

	P2	P3′	P4	P5	P6
q, i	−3,4	−4,2	−4,3	−5,1	−6,1
u_0	d^{-1}	$2c^{-1}x$	c^{-1}	$\theta_\infty^{-1}x$	$\theta_\infty^{-1}x(1-x)$
$\sqrt{\ }$	$d^2 = \delta$	$c^2 = \gamma$	$c^2 = \gamma$	$\theta_\infty^2 = 2\alpha$	$\theta_\infty^2 = 2\alpha$
α	$-d\theta_\infty$	$2c\theta_\infty$	$2c\theta_\infty$	$\theta_\infty^2/2$	$\theta_\infty^2/2$
β		$-2d\theta_0$	$-8\theta_0^2$	$-\theta_0^2/2$	$-\theta_0^2/2$
γ		c^2	c^2	$-d\theta_1$	$\theta_1^2/2$
δ	d^2	$-d^2$		$-d^2/2$	$(1-\theta_x^2)/2$
$\boldsymbol{\theta}$	(θ_∞)	$\begin{pmatrix}\theta_\infty\\ \theta_0\end{pmatrix}$	$\begin{pmatrix}\theta_\infty\\ \theta_0\end{pmatrix}$	$\begin{pmatrix}\theta_\infty\\ \theta_0\\ \theta_1\end{pmatrix}$	$\begin{pmatrix}\theta_\infty\\ \theta_0\\ \theta_1\\ \theta_x\end{pmatrix}$

the additional symbols γ in P4 and δ in P2 have been added to represent the signs of the two opposite residues $\pm u_0$ of each family of movable simple poles. The P3′ equation is that defined by Painlevé [1, p 1115] in the class of P3. Table 1 collects the relevant data, in particular the definition of the monodromy exponents.

Remark. Those of the parameters $\boldsymbol{\alpha}$ which are unused in the definition of the monodromy exponents $\boldsymbol{\theta}$ in table 1 are necessarily invariant under any birational transformation. Thus, $\Delta = \delta$ for $n = 5, 3, 2$, $\Gamma = \gamma$ for $n = 4, 3$.

The successive coalescences of the four singular points of P6 [1]

$$\begin{array}{ccccccc} & & & \mathrm{P4} & & & \\ & & \nearrow & & \searrow & & \\ \mathrm{P6} & \rightarrow & \mathrm{P5} & & & \mathrm{P2} & \\ & & \searrow & & \nearrow & & \\ & & & \mathrm{P3}' & & & \end{array} \tag{48}$$

from an equation $E(x, u, \alpha, \beta, \gamma, \delta) = 0$ to an equation $E(X, U, A, B, \Gamma, \Delta) = 0$ are described by Poincaré perturbations $(x, u, \alpha, \beta, \gamma, \delta, \boldsymbol{\theta}) \to (X, U, A, B, \Gamma, \Delta, \boldsymbol{\Theta}, \varepsilon)$, $\varepsilon \to 0$. The confluence formulae for $\alpha, \beta, \gamma, \delta$ are classical [1]; those for the monodromy exponents $\boldsymbol{\theta}$ have been established in [26], they are recalled in table 2. All these transformations are affine, and this is the reason why Painlevé introduced P3′ to replace his original choice of P3.

Let us first remark that the confluence cannot increase the degree of the birational transformation.

When acting on T_6, the confluence generates two inequivalent first-degree birational transformations at the P5 level. Indeed, at the P6 level the four singular points are equivalent but, for going to the P5 level (two equivalent singularities plus another one, chosen once and for all as $(\infty, 0)$ and 1), there exist two inequivalent choices of confluence, and both must be performed. The first choice, which we call 'normal', is the one adopted in table 2, which selects $(x, 1)$ as the coalescing pair. The second choice, called 'biased', is to select a pair made of one point among $(\infty, 0)$ and a second point among $(1, x)$. A given choice is equivalent to performing the corresponding homography prior to the standard

Table 2. Confluence of the monodromy exponents. The parameters c, d (which essentially represent signs) also participate to the confluence. The choice of square roots is such that there are only + signs in the successive values $\theta_\infty + \theta_0 + \theta_1 + \theta_x$, $\theta_\infty + \theta_0 + \theta_1$, $2\theta_\infty + 2\theta_0$, $\theta_\infty + \theta_0$, $2\theta_\infty$.

	x	u	θ_∞	θ_0	θ_1	θ_x	c	d
$6 \to 5$	$1 + \varepsilon X$	U	θ_∞	θ_0	$\theta_1 - \varepsilon^{-1} d$	$\varepsilon^{-1} d + \varepsilon$		
$5 \to 4$	$1 + \varepsilon X$	$\varepsilon U/2$	$-2c\varepsilon^{-2}$	$2\theta_0$	$2\theta_\infty + 2c\varepsilon^{-2}$			$2c\varepsilon^{-2} - 2\theta_\infty$
$5 \to 3'$	X	$1 + \varepsilon U$	$\theta_\infty + \varepsilon^{-1} c/2$	$-\varepsilon^{-1} c/2$	θ_0			$\varepsilon d/2$
$4 \to 2$	$\varepsilon X/4 - \varepsilon^{-1}$	$\varepsilon^{-1} + U$	$-\varepsilon^{-3} d$	$\theta_\infty + \varepsilon^{-3} d$			$4\varepsilon^{-1} d$	
$3' \to 2$	$1 + \varepsilon^2 X/2$	$1 + \varepsilon U$	$4d\varepsilon^{-3}$	$2\theta_\infty - 4d\varepsilon^{-3}$			$-4d\varepsilon^{-3}$	$2\theta_\infty + 4d\varepsilon^{-3}$

confluence. Among the four homographies which conserve x and P6, two (the identity and H_{badc}) correspond to the first choice and two (H_{dcba} and H_{cdab}) to the second choice. To maximize the symmetry, let us choose the biased transformation $\mathrm{T}_{6,b}$ as the product of T_6 on the right by H_{dcba}, with signs reversals for the ∞ and x components (S_a, S_b, S_c, S_d denote the operators which change the sign of, respectively, θ_∞, θ_0, θ_1, θ_x),

$$\mathrm{T}_{6,b} = S_a S_d \mathrm{T}_6 \mathrm{H}_{dcba} S_d S_a. \tag{49}$$

Its affine and birational representations are

$$\mathrm{P6}\ :\ \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \\ s_x\theta_x \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \\ S_1\Theta_1 \\ S_x\Theta_x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \tag{50}$$

and

$$\begin{aligned} &\frac{-Nx(x-1)}{(u-x)(U-x) - x(x-1)} \\ &\qquad = (U-x)\left(\frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_\infty - \Theta_0 - \Theta_1}{U-x}\right) \end{aligned} \tag{51}$$

$$\frac{Nx(x-1)}{(u-x)(U-x) - x(x-1)} = (u-x)\left(\frac{x(x-1)u'}{u(u-1)(u-x)} + \frac{\theta_0}{u} + \frac{\theta_1}{u-1} + \frac{\theta_\infty - \theta_0 - \theta_1}{u-x}\right) \tag{52}$$

in which the odd-parity constant N, again equal to the difference of the squared norms of the monodromy exponents, as in equation (36), now takes different affine expressions in (θ_j, Θ_j),

$$N = \sum(\theta_k^2 - \Theta_k^2) \tag{53}$$

$$= 1 + S_\infty\Theta_\infty - S_0\Theta_0 - S_1\Theta_1 + S_x\Theta_x \tag{54}$$

$$= -1 - s_\infty\theta_\infty + s_0\theta_0 + s_1\theta_1 - s_x\theta_x \tag{55}$$

$$\begin{aligned} &= -2(s_\infty\theta_\infty - S_x\Theta_x) = 2(s_0\theta_0 - S_1\Theta_1) = 2(s_1\theta_1 - S_0\Theta_0) \\ &= -2(s_x\theta_x - S_\infty\Theta_\infty). \end{aligned} \tag{56}$$

Therefore, under the standard confluence, T_6 and $\mathrm{T}_{6,b}$ generate two *inequivalent* first-degree birational transformations at the P5 level. The sum $\theta_\infty + \theta_0 + \theta_1 + \theta_x$ goes to either $\theta_\infty + \theta_0 + \theta_1$ or to $2d$, depending on the product of the signs of θ_1 and θ_x, see table 2.

The process could continue for P5 $\to$ P4 or others, but this does not happen. Indeed, this noncommutativity of the homographies and the confluence never occurs at a lower level, since it is already taken into account by the fact that P5 has two daughter Pn equations.

The two sequences of first-degree birational transformations (normal and biased) are described in the next two sections. The monodromy exponents θ_j are defined in table 1. By convention, when the signs satisfy $s_j = S_j$, the birational transformation is equal to its inverse.

Let us now perform the standard confluence on T_6 and $\mathrm{T}_{6,b}$, both for the affine representation and the birational representation (39). This will allow us to exhaust all the first-degree birational transformations found by various authors.

On the affine matrix representation such as (50), the confluence acts as

$$\begin{array}{ll} 6 \to 5 : \text{lines } (1, 2, 3+4) \to (1, 2, 3) & \text{columns } (1, 2, 3) \to (1, 2, 3) \\ 5 \to 4 : \text{lines } (1+3, 2) \to (1, 2) & \text{columns } (1, 2) \to (1, 2) \\ 5 \to 3 : \text{lines } (1+2, 3) \to (1, 2) & \text{columns } (1, 3) \to (1, 2) \\ 4 \to 2 : \text{lines } (1+2) \to (1) & \text{columns } (1) \to (1) \\ 3 \to 2 : \text{lines } (2) \to (1) & \text{columns } (1) \to (1). \end{array} \tag{57}$$

6.1. The normal sequence

This sequence is

$$\mathrm{P6}\ :\ \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \\ s_x\theta_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \\ S_1\Theta_1 \\ S_x\Theta_x \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{58}$$

$$\mathrm{P5}\ :\ \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -2 & -2 & 0 \end{pmatrix}\begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \\ S_1\Theta_1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad s_\infty d = S_\infty D \tag{59}$$

$$\mathrm{P4}\ :\ \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad s_\infty c = S_\infty C \tag{60}$$

$$\mathrm{P3}'\ :\ \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad s_\infty c = S_\infty C \quad s_0 d = S_0 D \tag{61}$$

$$\mathrm{P2}\ :\ (s\theta_\infty) = -(S\Theta_\infty) + (1) \qquad s_\infty d = S_\infty D. \tag{62}$$

For the signs $s_j = 1$, all the shifts are positive, and, for the signs $s_j = S_j$, the linear part has determinant -1. The sum of the shifts remains equal to two (except for P4 and P2 because of a global rescaling, see table 1).

Since the affine representations (58)–(62) are involutions when the signs satisfy $s_j = S_j$, only half of the birational representation needs to be written, the second half resulting from the permutation of $(u, \boldsymbol{\theta}, c, d)$ and $(U, \boldsymbol{\Theta}, C, D)$.

The confluence on T_6 results in (we omit the signs, they can easily be restored)

$$\mathrm{P6}\ :\ \frac{N}{u-U} = \frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_x - 1}{U-x} \tag{63}$$

$$\mathrm{P5}\ :\ \frac{N}{u-U} = \frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_1 - 1}{U-1} + \frac{Dx}{(U-1)^2} \tag{64}$$

$$\mathrm{P4}\ :\ \frac{N}{u-U} = \frac{U'}{U} + \frac{4\Theta_0}{U} + CU + 2Cx \tag{65}$$

$$\mathrm{P3}'\ :\ \frac{N}{u-U} = \frac{xU'}{U^2} + \frac{\Theta_0 - 1}{U} + \frac{Dx}{2U^2} - \frac{C}{2} \tag{66}$$

$$\mathrm{P2}\ :\ \frac{N}{u-U} = U' + DU^2 + D\frac{x}{2} \tag{67}$$

with

$$\text{P6} \ : \ N = 1 - \Theta_\infty - \Theta_0 - \Theta_1 - \Theta_x = \left(\tfrac{1}{2}\right) \sum (\theta_j - \Theta_j) \tag{68}$$

$$\text{P5} \ : \ N = 1 - \Theta_\infty - \Theta_0 - \Theta_1 = \left(\tfrac{1}{2}\right) \sum (\theta_j - \Theta_j) \tag{69}$$

$$\text{P4} \ : \ N = -2(1 - 2\Theta_\infty - 2\Theta_0) = 2 \sum (\theta_j - \Theta_j) \tag{70}$$

$$\text{P3}' \ : \ N = 1 - \Theta_\infty - \Theta_0 = \left(\tfrac{1}{2}\right) \sum (\theta_j - \Theta_j) \tag{71}$$

$$\text{P2} \ : \ N = \tfrac{1}{2} - \Theta_\infty = \left(\tfrac{1}{2}\right) (\theta_\infty - \Theta_\infty). \tag{72}$$

These transformations were first found respectively, for P5 by Okamoto [28], for P4 by Murata [29], for P3 by Fokas and Ablowitz [17, equations (4.4*a*), (4.4*b*)], for P2 by Lukashevich [30].

6.2. *The biased sequence*

On the biased transformation $\mathrm{T}_{6,b}$ (also an involution), the confluence yields

$$\text{P6}: \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \\ s_x\theta_x \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \\ S_1\Theta_1 \\ S_x\Theta_x \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \tag{73}$$

$$\text{P5}: \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \\ s_1\theta_1 \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \\ S_1\Theta_1 \end{pmatrix} + \tfrac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad s_\infty d = -S_\infty D \tag{74}$$

$$\text{P4}: \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \end{pmatrix} = -\tfrac{1}{2} \begin{pmatrix} -1 & -3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \end{pmatrix} + \tfrac{1}{4} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad s_\infty c = -S_\infty C \tag{75}$$

$$\text{P3}': \begin{pmatrix} s_\infty\theta_\infty \\ s_0\theta_0 \end{pmatrix} = - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} S_\infty\Theta_\infty \\ S_0\Theta_0 \end{pmatrix} \qquad s_\infty c = -S_\infty C \quad s_0 d = -S_0 D \tag{76}$$

$$\text{P2}: (\, s\theta_\infty \,) = (\, S\Theta_\infty \,) \qquad s_\infty d = -S_\infty D. \tag{77}$$

At the P3 level, the transformation reduces to the permutation of the two singular points $(\infty, 0)$, a homography on u which leaves P3 invariant. Therefore, at the P2 level this is just the identity.

The biased affine representations (73)–(77) and the normal ones (58)–(62) have opposite linear parts (this results from our involution convention), but the sum of the biased shifts is zero.

The biased birational transformations are

$$\text{P6}: \frac{-Nx(x-1)}{(u-x)(U-x) - x(x-1)} = (U - x) \left(\frac{x(x-1)U'}{U(U-1)(U-x)} + \frac{\Theta_0}{U} + \frac{\Theta_1}{U-1} + \frac{\Theta_\infty - \Theta_0 - \Theta_1}{U-x} \right) \tag{78}$$

$$\text{P5}: \frac{-2Dx}{(u-1)(U-1)} = (U-1) \left(\frac{xU'}{U(U-1)^2} + \frac{\Theta_0}{U} + \frac{\Theta_\infty - \Theta_0}{U-1} + \frac{Dx}{(U-1)^2} \right) \tag{79}$$

$$\text{P4}: 2C(u+U) = \frac{U'}{U} + \frac{4\Theta_0}{U} + CU - 2Cx \tag{80}$$

$$\text{P3}: \frac{Dx}{uU} = -C \tag{81}$$

$$\text{P2}: u + U = 0. \tag{82}$$

In (78), the constant N is any expression among (53)–(56). The transformation for P5 was first obtained by Gromak [31, equation (13)], and the one for P4 by Lukashevich [32].

Let us denote $\mathrm{T}_{6,b}$, $\mathrm{T}_{5,b}$ and $\mathrm{T}_{4,b}$ the biased transformations, $\mathrm{T}_{6,u}$, $\mathrm{T}_{5,u}$ and $\mathrm{T}_{4,u}$ the normal ones, and H the unique homography of P5 which conserves x,

$$\mathrm{P5}: \mathrm{H}(x, u, \theta_\infty, \theta_0, \theta_1) = (x, u^{-1}, \theta_0, \theta_\infty, \theta_1). \tag{83}$$

One has the relations

$$\mathrm{T}_{6,u} = S_a S_d \mathrm{T}_{6,b} \mathrm{H}_{dcba} S_d S_a \tag{84}$$

$$\mathrm{T}_{5,u} = S_a \mathrm{T}_{5,b} S_a S_c \mathrm{T}_{5,b} S_a \mathrm{H} \tag{85}$$

$$\mathrm{T}_{4,u} = S_a \mathrm{T}_{4,b} S_b \mathrm{T}_{4,b} S_a \tag{86}$$

(the relation (86) is due to [33]). At the P6 level, the relation is clearly invertible but, at the P5 and P4 levels, we could not find inverse relations expressing the biased transformations as powers of the unbiased ones. Therefore, the biased birational transformations are more elementary than the unbiased ones.

7. Contiguity relation and its continuum limit

From each birational transformation, one easily deduces a contiguity relation, which generalizes, as noted by Garnier [5], that of the hypergeometric equation of Gauss. Its systematic computation is as follows [34].

(1) Consider the birational transformation, i.e. the direct birational transformation and its inverse

$$u = f(U, U', x, \boldsymbol{\theta}, \boldsymbol{\Theta}) \qquad \boldsymbol{\theta} = g(\boldsymbol{\Theta}) \tag{87}$$

$$U = F(u, u', x, \boldsymbol{\Theta}, \boldsymbol{\theta}) \qquad \boldsymbol{\Theta} = G(\boldsymbol{\theta}). \tag{88}$$

(2) Evaluate it at the values $(\underline{v}, v, \bar{v})$ taken by a discrete variable at three contiguous points $(z-h, z, z+h)$, with $z = nh$,

$$\bar{v} = f(v, v', x, f(\boldsymbol{\theta}), \boldsymbol{\theta}) \tag{89}$$

$$\underline{v} = F(v, v', x, F(\boldsymbol{\theta}), \boldsymbol{\theta}). \tag{90}$$

(3) Eliminate the variable v' between these two relations,

$$G(\bar{v}, \underline{v}, v, x, \boldsymbol{\theta}) = 0. \tag{91}$$

For P6, equations (87), (88) are equivalent to

$$\frac{x(x-1)U'}{U(U-1)(U-x)} = 2\frac{s_j\theta_j - S_j\Theta_j}{u-U} - \left(\frac{S_0\Theta_0}{U} + \frac{S_1\Theta_1}{U-1} + \frac{S_x\Theta_x - 1}{U-x}\right) \tag{92}$$

$$\frac{x(x-1)u'}{u(u-1)(u-x)} = -2\frac{S_j\Theta_j - s_j\theta_j}{u-U} - \left(\frac{s_0\theta_0}{u} + \frac{s_1\theta_1}{u-1} + \frac{s_x\theta_x - 1}{u-x}\right) \tag{93}$$

in which j is any one of the four singular points $(\infty, 0, 1, x)$, and the contiguity relation is

$$\frac{\varphi\left(n+\frac{1}{2}\right)}{\bar{v}-v} + \frac{\varphi\left(n-\frac{1}{2}\right)}{\underline{v}-v} = \frac{s_0\theta_0 - S_0\Theta_0}{v} + \frac{s_1\theta_1 - S_1\Theta_1}{v-1} + \frac{s_x\theta_x - S_x\Theta_x}{v-x} \tag{94}$$

$$\varphi(n) = \tfrac{1}{2}(s_\infty\theta_\infty + s_0\theta_0 + s_1\theta_1 + s_x\theta_x - 1) \tag{95}$$

in which $\boldsymbol{\theta}$ is taken at the centre point $z = z_0 + nh$. This very simple expression is clearly invariant under any permutation of the four singular points of P6.

This contiguity relation (94) can be interpreted as a second-order discrete equation [34]. The two-point recurrence relation (34) admits five classes of solutions. Each class, characterized by a signature, leads to a different contiguity relation (94), i.e. to a different second-order discrete equation. Four of them are autonomous (signatures $(s_j S_j) = (- - -+)$, $(- - ++)$, $(- + ++)$, $(+ + ++)$), they cannot admit a continuum limit to a Painlevé equation. The only nonautonomous one (signature $(- - - -)$) is

$$\frac{n+\frac{1}{2}}{\bar{v}-v} + \frac{n-\frac{1}{2}}{\underline{v}-v} = \frac{n+K_2(-1)^n}{v} + \frac{n+K_3(-1)^n}{v-1} + \frac{n+K_4(-1)^n}{v-x} \tag{96}$$

$$K_2 = -k_2 + k_3 + k_4 \qquad K_3 = k_2 - k_3 + k_4 \qquad K_4 = k_2 + k_3 - k_4. \tag{97}$$

In the continuum limit, among the six simple poles of v in the sum (including ∞), the first two will create a second-order derivative and the four others will define at most four singular points. Since none of the last four poles depends on n, it is impossible that the continuum limit be P6.

The transform of this discrete equation under

$$(\bar{v}, v, \underline{v}) \mapsto (\bar{v}, x/v, \underline{v}) \tag{98}$$

has already been obtained [35, equation (1.5)] as a reduction of a lattice KdV equation, together with a discrete Lax pair and a continuum limit to the full P5. This is in agreement with the continuum limit of the hypergeometric contiguity relation, which is not the hypergeometric equation but a confluent one. Nevertheless, we do not know of a general proof of this feature.

8. Conclusion

The fundamental homography between the derivative of the solution of the Painlevé equation and the Riccati pseudopotential has allowed us to define a consistent truncation. For P6, this truncation provides one first-degree birational transformation. Under the confluence to the lower P*n* equations, this transformation generates two inequivalent first-degree birational transformations at the P5 and P4 level, and only one at the P3 and P2 level. This provides a coherent description of all the first-degree birational transformations of all the P*n* equations which admit one.

As to the contiguity relation of the first-degree birational transformation of P6, it defines exactly one nonautonomous second-order difference equation. If again one applies to this single discrete equation the two noncommuting operations of confluence and homography which have generated the normal and biased sequences of birational transformations, one should similarly define a coherent description including many of the currently existing discrete Painlevé equations.

Acknowledgments

The authors warmly thank J Satsuma and T Tokihiro for their generous hospitality, and they acknowledge the financial support of the Tournesol grant T99/040. RC thanks the French ‘ministère des affaires étrangères’ for travel support.

References

[1] Painlevé P 1906 Sur les équations différentielles du second ordre à points critiques fixes *C. R. Acad. Sci., Paris* **143** 1111–17
[2] Schlesinger L 1912 Über eine Klasse von differentialsystemen beliebiger Ordnung mit festen kritischen Punkten *J. Fürr. Angew. Math.* **141** 96–145
[3] Fuchs R 1905 Sur quelques équations différentielles linéaires du second ordre *C. R. Acad. Sci., Paris* **141** 555–8
[4] Garnier R 1943 Sur l’existence de relations entre des fonctions contiguës de Painlevé *C. R. Acad. Sci., Paris* **217** 60–2
[5] Garnier R 1951 Sur un théorème de Schwarz *Commun. Math. Helv.* **25** 140–72

[6] Fokas A S and Yortsos Y C 1981 The transformation properties of the sixth Painlevé equation and one-parameter families of solutions *Lett. Nuovo Cimento* **30** 539–44

[7] Okamoto K 1987 Studies on the Painlevé equations: I. Sixth Painlevé equation *Ann. Mat. Pura Appl.* **146** 337–81

[8] Muğan U and Sakka A 1995 Schlesinger transformations for Painlevé VI equation *J. Math. Phys.* **36** 1284–98

[9] Malmquist J 1922–23 Sur les équations différentielles du second ordre dont l'intégrale générale a ses points critiques fixes *Arkiv Math. Astron. Fys.* **17** 1–89

[10] Conte R 2001 Sur les transformations de Schlesinger de la sixième équation de Painlevé *C. R. Acad. Sci., Paris* **332** 501–4
(Conte R 2001 *Preprint* math.CA/0103165)

[11] Nijhoff F W, Joshi N and Hone A 2000 On the discrete and continuous Miura chain associated with the sixth Painlevé equation *Phys. Lett.* A **264** 396–406

[12] Weiss J, Tabor M and Carnevale G 1983 The Painlevé property for partial differential equations *J. Math. Phys.* **24** 522–6

[13] Musette M 1999 Painlevé analysis for nonlinear partial differential equations *The Painlevé Property, One Century Later (CRM Series in Mathematical Physics)* ed R Conte (Berlin: Springer) pp 517–72

[14] Conte R 2001 Exact solutions of nonlinear partial differential equations by singularity analysis *Direct and Inverse Methods in Nonlinear Evolution Equations* ed A Greco (Berlin: Springer) p 83
(Conte R 2000 *Preprint* nlin-SI/0009024 CIME school, Cetraro)

[15] Clarkson P A, Joshi N and Pickering A 1999 Bäcklund transformations for the second Painlevé hierarchy: a modified truncation approach *Inverse Problems* **15** 175–87

[16] Gordoa P, Joshi N and Pickering A 1999 Mappings preserving locations of movable poles: a new extension of the truncation method to ordinary differential equations *Nonlinearity* **12** 955–68

[17] Fokas A S and Ablowitz M J 1982 On a unified approach to transformations and elementary solutions of Painlevé equations *J. Math. Phys.* **23** 2033–42

[18] Conte R and Musette M 2001 New contiguity relation of the sixth Painlevé equation from a truncation *Preprint* S2001/009 *Physica* D, at press

[19] Musette M and Conte R 1991 Algorithmic method for deriving Lax pairs from the invariant Painlevé analysis of nonlinear partial differential equations *J. Math. Phys.* **32** 1450–7

[20] Painlevé P 1897 *Leçons sur la théorie analytique des équations différentielles (Leçons de Stockholm, 1895)* (Paris: Hermann) (1973 *Œuvres de Paul Painlevé* vol 1 (Paris: Éditions du CNRS) (Reprint))

[21] Gromak V I and Lukashevich N A 1990 *The Analytic Solutions of the Painlevé Equations* (Minsk: Universitetskoye Publishers) p 157 ISBN 5–7855–0319–0 (in Russian)

[22] Hone A N W 1998 Nonautonomous Hénon–Heiles systems *Physica* D **118** 1–16

[23] Mazzocco M 2001 Picard and Chazy solutions to the Painlevé-VI equation *Math. Ann.* **321** 131–69
(Mazzocco M 1999 *Preprint* math.AG/9901054)

[24] Umemura H 1996 Special polynomials associated with the Painlevé equations, I *Theory of Nonlinear Special Functions: the Painlevé Transcendents (Montréal Workshop) (May 1996)* ed L Vinet and P Winternitz

[25] Gambier B 1910 Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est à points critiques fixes 1910 *Acta Math.* **33** 1–55

[26] Okamoto K 1986 Isomonodromic deformation and Painlevé equations, and the Garnier system *J. Fac. Sci. Univ. Tokyo* IA **33** 575–618

[27] Jimbo M and Miwa T 1981 Monodromy preserving deformations of linear ordinary differential equations with rational coefficients II *Physica* D **2** 407–48

[28] Okamoto K 1987 Studies on the Painlevé equations: II. Fifth Painlevé equation *Japan. J. Math.* **13** 47–76

[29] Murata Y 1985 Rational solutions of the second and the fourth equations of Painlevé *Funk. Ekvacioj. Ser. Int.* **28** 1–32

[30] Lukashevich N A 1971 The second Painlevé equation *Diff. Urav.* **7** 1124–5 (Engl. Transl. 1971 *Diff. Eqns* **7** 853–4)

[31] Gromak V I 1976 Solutions of Painlevé's fifth problem *Diff. Urav.* **12** 740–6 (Engl. Transl. 1976 *Diff. Eqns* **12** 519–21)

[32] Lukashevich N A 1967 Theory of the fourth Painlevé equation *Diff. Urav.* **3** 771–89 (Engl. Transl. 1967 *Diff. Eqns* **3** 395–9)

[33] Bassom A P, Clarkson P A and Hicks A C 1995 Bäcklund transformations and solution hierarchies for the fourth Painlevé equation *Stud. Appl. Math.* **95** 1–71

[34] Fokas A S, Grammaticos B and Ramani A 1993 From continuous to discrete Painlevé equations *J. Math. Anal. Appl.* **180** 342–60

[35] Nijhoff F W, Ramani A, Grammaticos B and Ohta Y 2001 On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation *Stud. Appl. Math.* **106** 261–314
(Nijhoff F W, Ramani A, Grammaticos B and Ohta Y 1998 *Preprint* solv-int/9812011)

[36] Gordoa P, Joshi N and Pickering A 2001 Mappings preserving locations of movable poles: II. The third and fifth Painlevé equations *Nonlinearity* **14** 567–82